CHAPTER 3
Working with pressure and stress

☞ Quantify the effect of pressure and viscous stress on and in fluids

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last modified: 2026-11-24

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3.1 What are we doing here?

Only three types of forces determine where fluids are going: gravity, pressure and viscous stress. Gravity is easy, and here we want to explore the other two. What *are* pressure and viscous stress actually? What do they *do* to fluids?

- we want to have extremely clear ideas about *how* pressure and viscous stress are expressed;
- we want to be able to calculate forces exerted by fluids on solid objects.

3.2 Pressure force on solid surfaces

You can easily imagine that if a fluid uniformly exerts a pressure p on a solid, flat surface with area S, the resulting force will be $F_{\text{pressure}} = pS$, pushing in the direction normal to the surface.

Things get more interesting when we consider cases where the pressure exerted by the fluid is not uniform, like in Figure 3.1.

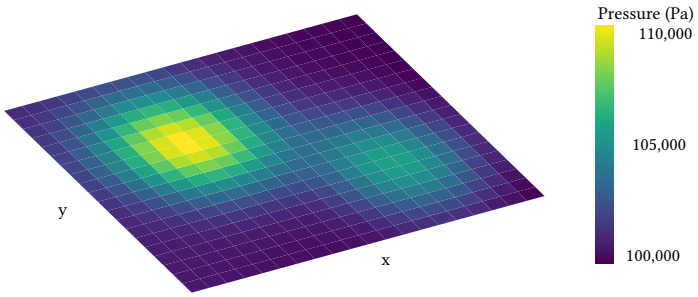


Figure 3.1: In this simple, made-up case, the fluid exerts pressure on a solid square-shaped object; the pressure is not uniform.

Diagram by OC

In this new case we obtain the force by integration (a fancy way of carrying out a "weighted sum"). The surface is split into infinitesimal chunks of area $\mathrm{d}S$, and the corresponding forces are summed up as:

$$F_{\text{pressure}} = \int_S \mathrm{d}F_{\text{pressure}} = \int_S p \, \mathrm{d}S \tag{3/1}$$

for a flat surface.
where the S-integral denotes an integration over the entire surface.

What we need to calculate the force in equation 3/1 is an expression of p as a function of the position on the surface S. When the fluid is static, this will be relatively easy. In two dimensions x and y, we rewrite $\mathrm{d}S$ as $\mathrm{d}S = \mathrm{d}x \, \mathrm{d}y$ and we may then proceed with the calculation starting from

$$F_{\text{pressure}} = \iint p_{(x,y)} \, \mathrm{d}x \, \mathrm{d}y \tag{3/2}$$

A computer will readily compute this for *any* non-uniform pressure, but in this course we will do hand calculations only for cases where the pressure varies in a manageable way, as shown in Figure 3.2. We will get to these further down.

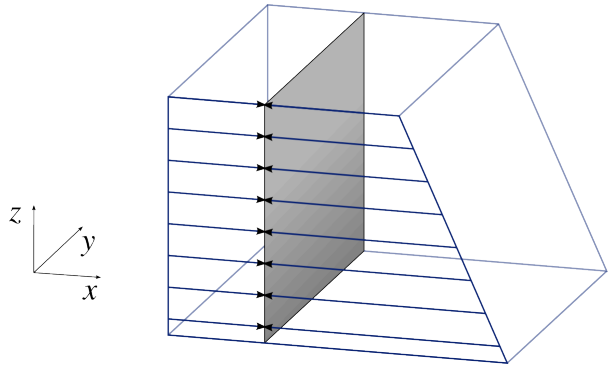


Figure 3.2: Pressure distribution on a flat plate. We already studied this situation in the problem sheet for chapter 1.

Figure CC-0 OC

Where does the force act? To find out, we pick a hinge axis anywhere in the plane of the surface, and we calculate the moment M_{axis} exerted by pressure about the axis. Each infinitesimal force $\mathrm{d}F$ acts at a distance $r_{\text{axis}F}$ from the axis, and so

$$M_{\text{axis}} = \int_S \mathrm{d}M_{\text{axis}} = \int_S r_{\text{axis}F}\ \mathrm{d}F = \int_S r_{\text{axis}F}\ p\ \ \mathrm{d}S \tag{3/3}$$

for a flat surface, with the hinge axis in the plane of the surface.

Once we have both F_{pressure} and M_{axis}, we can calculate the distance $R_{\text{axis}\,F}$ between the hinge axis and the application point of the net pressure force:

$$R_{\text{axis}F} = \frac{M_{\text{axis}}}{F_{\text{pressure}}} \tag{3/4}$$

Of course, not every object has flat surfaces. Once the surface becomes curved, as in Figure 3.3, we need to carry out the operation using vectors:

$$\vec{F}_{\text{pressure}} = \int_S \mathrm{d}\vec{F} = \int_S -p\,\vec{n}\ \mathrm{d}S \tag{3/5}$$

where S indicates an integration over the entire surface;
and $\vec{n}$ is a unit vector describing, on each infinitesimal surface element $\mathrm{d}S$, the direction normal to the surface.

The position of the force vector can also be determined with one moment integral, but this is outside the scope of this course.

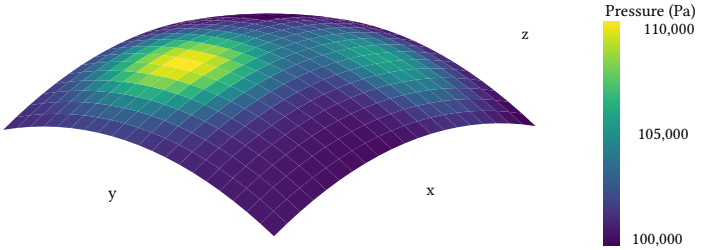


Figure 3.3: On a curved solid surface, the pressure force *direction* would need to be evaluated on each element too. The sum of all the small pressure force vectors would be the net pressure force.

Diagram by OC

An important point is that regardless of the shape of the solid surface, the local pressure results in a local force perpendicular to it. Pressure on each element only has one value:

> Pressure at a point on a solid surface only has one value (in $\mathrm{N/m^2}$).

When they flow, fluids exert pressure on solid surfaces that show up in all kinds of interesting patterns. Take some time to observe Figure 3.4, looking for places of high and low pressure on the walls of the channel.

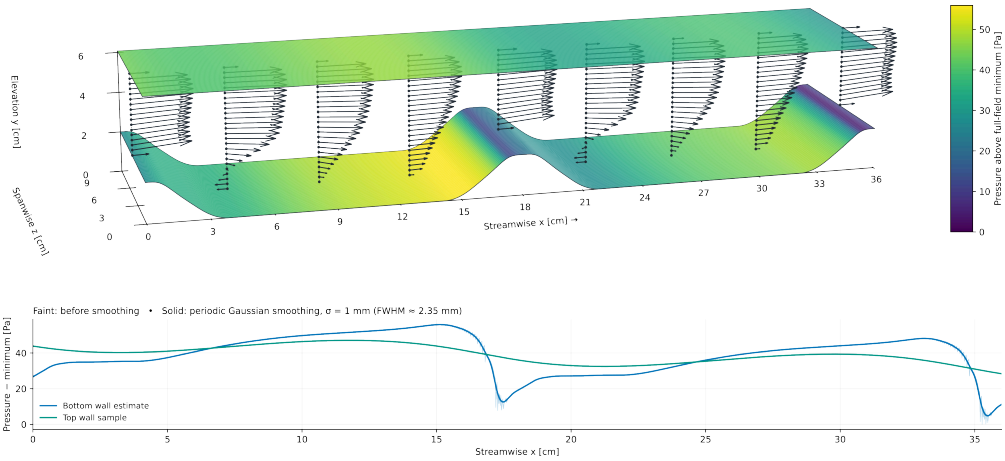


Figure 3.4: In this flow, water flows from left to right in a channel with a flat (solid) ceiling and a wavy floor. The outlet is fed back into the inlet (a technique called periodic boundary conditions) to simulate the case of an infinitely long channel. The distance between hills is 18 cm and the nominal speed is 0,28 $\mathrm{m\,s^{-1}}$. There is no gravity. The Reynolds number is [Re] = 5600; the flow is mildly turbulent.

This visualization is extracted from a high-fidelity (Direct Numerical Simulation or DNS) simulation, and is time-averaged. The pressure "zero" is arbitrarily chosen at the lowest pressure value. Two grooves are shown, but only one is simulated and the resulting data is simply repeated.

Diagram by OC, built with GPT-6 Astra, based on source data kindly provided by Sylvain Laizet and Xuhui Zhou [8, 11]

A few representative values for pressure, given to help build your sense of what is a "big" or "small" pressure, are given in Figure 3.5. However, when it comes to moving fluids, it is not the pressures but the pressure *differences* that matter. Pressure differences are a big part of what drives the flow and influences where the fluid is going. Indicative pressure differences for a few flows are given in Figure 3.6.

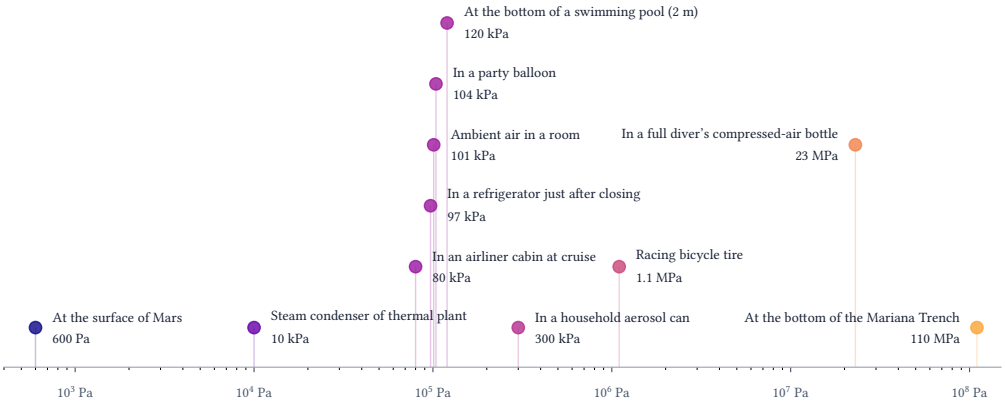


Figure 3.5: A few representative pressures. Be cautious while reading this diagram: the scale is logarithmic.

Diagram by OC

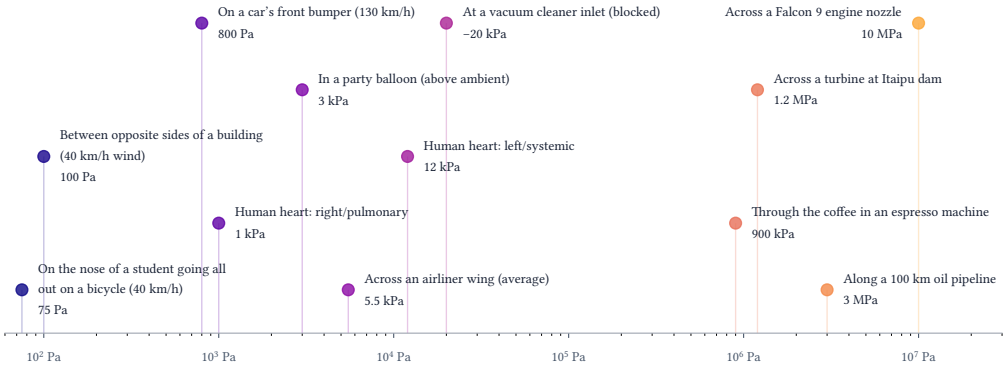


Figure 3.6: A few representative pressure differences. When not explicit, the difference is to ambient conditions; only the absolute value is plotted. Be cautious while reading this diagram: the scale is logarithmic.

Diagram by OC

3.3 Pressure force within fluids

Fluids don't just exert pressure on solid objects: they also exert pressure within themselves. Let's explore.

3.3.1 Pressure at a point

The first important fact to have in mind is that **pressure has no direction**:

> Pressure at a point is *one number* (in $\mathrm{N/m^2}$).

Another way of formulating this is that pressure in a fluid is a scalar field. Pressure is not the same everywhere, but at any one point, it has only one value, the same in every direction. Figure 3.7 shows the pressure field in one flow. Do spend some time observing where pressure is high or low. In general, the flow turns away from areas with high pressure, and towards areas with low pressure.

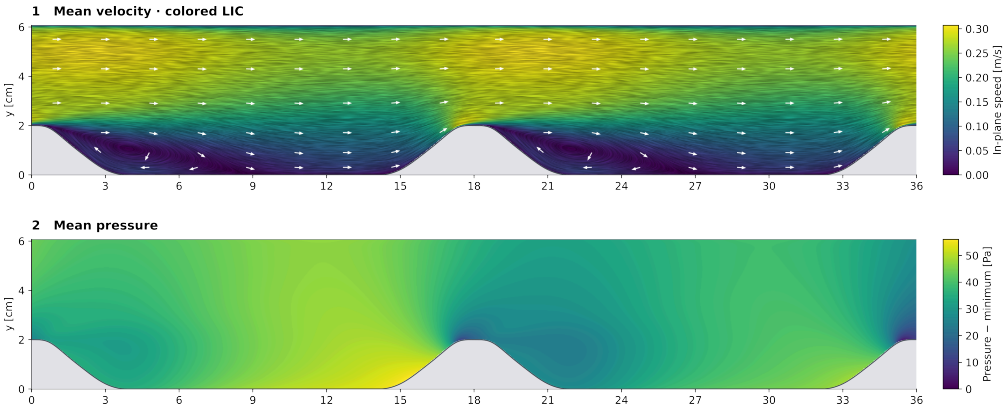


Figure 3.7: The velocity (top) and pressure (bottom) in the flow from Figure 3.4. The data is time-averaged, and displayed in a flow-wise plane in the middle of the channel.

The velocity has not just magnitude, but direction; the direction is indicated with line integral convolution patterns (LIC) and with white vector arrows. Due to the time-averaging, smooth patterns do not mean that the instantaneous flow is laminar.

Diagram by OC, built with GPT-6 Astra, based on source data kindly provided by Sylvain Laizet and Xuhui Zhou [8, 11]

Why does fluid respond this way to pressure? Or more precisely: how does pressure influence the flow? We need to explore further.

Advice from an expert



Take all the time you need to read into Figure 3.7, which is a good opportunity to unlearn the Bernoulli equation if needed. Yes, in a very loose way, high velocity correlates with low pressure and vice-versa. But look closely: many things are happening which disprove a direct relationship: for example, where the flow separates from the hill and recirculates. Also, the velocity is (low) zero at the top wall, but this does not show on the pressure plot. Likewise, the pressure keeps decreasing all along the flow, but the bulk velocity remains the ame. The effects we neglected when we wrote the Bernoulli equation in chapter 2 *also* play a role in determining the velocity — not just pressure!

3.3.2 Pressure traction on a surface

Imagine cutting out an infinitesimal cube of fluid from inside an interesting flow. Each of its six faces is at a different position, and so may be experiencing a different effort (Figure 3.8). We call each of those the *pressure traction on a surface*. It is a vector representing, in $\mathrm{N/m^2}$, the force per unit area exerted on the chosen side of that surface. Pressure traction is always perpendicular to the surface.

$$\vec{t}_p \equiv -p\vec{n} \tag{3/6}$$

where $\vec{n}$ is a unit vector pointing outwards of the chosen surface.

Pressure traction across a surface is a vector (in $\mathrm{N/m^2}$), always normal to that surface.

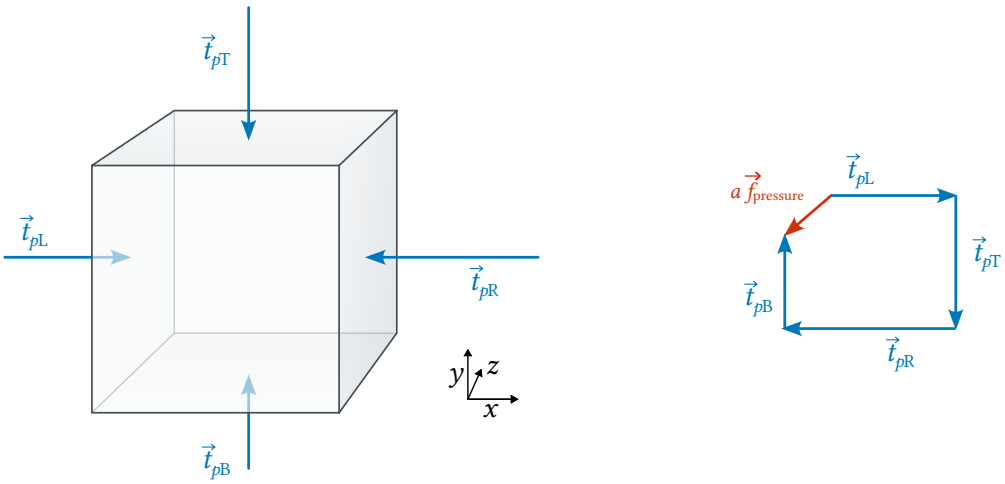


Figure 3.8: Four sides of an infinitesimal cube experience four different pressure tractions. The sum of these tractions, divided by the side length a of the cube, is the pressure force per unit volume $\vec{f}_{\text{pressure}}$. The front and back sides are left empty here for clarity, but they experience pressure tractions too, and contribute to the vector sum.

Diagram by OC

3.3.3 Pressure force per unit volume

The six tractions on the faces of the cube, all together, will result in a net force per unit volume. This force can point in any direction. It expresses the net effect of pressure on that cube.

The pressure force per unit volume is a vector (in $\mathrm{N/m^3}$), pointing in any direction

How does it relate to pressure exactly? Let's examine the cube again, in Figure 3.9. We can express the differences between each pair of faces with small differentials $\mathrm{d}p$.

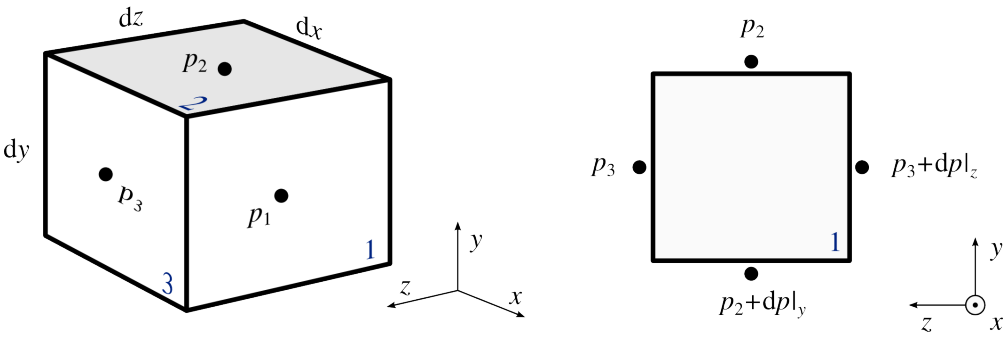


Figure 3.9: Each face of a tiny (infinitesimal) cube experiences a different pressure. The differences are labeled $\mathrm{d}p|_i$ in each of the $i = x, y, z$ directions.

figure CC-0 OC

We begin with the x-direction. The only relevant tractions are those on faces 1 and 4. Each acts upon an area $\mathrm{d}y\ \mathrm{d}z$:

$$F_{\text{net, pressure},x} = \mathrm{d}S\left(t_{p\mathrm{L}} - t_{p\mathrm{R}}\right) = \mathrm{d}y\ \mathrm{d}z[p_1 - p_4]$$

Since the cube is infinitesimal, we express $p_4 - p_1$ as the derivative of pressure in the x-direction ($\partial p/\partial x$) multiplied by the distance $\mathrm{d}x$ which separates points 1 and 4, obtaining:

$$F_{\text{net, pressure},x} = \mathrm{d}y\ \mathrm{d}z\left[-\frac{\partial\,\mathrm{p}}{\partial\,\mathrm{x}}\ \mathrm{d}x\right]$$

$$= \mathrm{d}\mathscr{V}\frac{-\partial p}{\partial x}$$

where $\mathrm{d}\mathscr{V} \equiv \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z$ is the volume of the infinitesimal cube.

From here, it is not too hard to write the force components in the other two directions:

$$F_{\text{net, pressure},x} = \mathrm{d}\mathscr{V}\frac{-\partial p}{\partial x}$$

$$F_{\text{net, pressure},y} = \mathrm{d}\mathscr{V}\frac{-\partial p}{\partial y}$$

$$F_{\text{net, pressure},z} = \mathrm{d}\mathscr{V}\frac{-\partial p}{\partial z}$$

This is tedious to write, but we recognize a pattern. (Mathematicians are fond of patterns.) To make things sharp and concise, we introduce the concept of *gradient*, a mathematical operator, defined as follows:

$$\vec{\nabla} \equiv \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \tag{3/7}$$

With this cool new tool, we elegantly rewrite the group of equations above:

$$\vec{F}_{\text{net, pressure}} = -\ \mathrm{d}\mathscr{V}\,\vec{\nabla}p$$

Finally, we obtain:

$$\frac{1}{\mathrm{d}\mathscr{V}}\vec{F}_{\text{net, pressure}} = -\vec{\nabla}p \tag{3/8}$$

This last equation reads "the pressure force per unit volume is the opposite of the pressure gradient". It shows us that in any fluid and any situation, the force due to pressure points in the direction opposite to the pressure gradient. If we follow one particle inside a fluid flow, we can quantify in which direction, and with which magnitude, pressure (a *scalar* field) is going to "push" it. This is given by equation 3/8, which quantifies this effect as a *vector* field. One such vector field is shown in Figure 3.10.

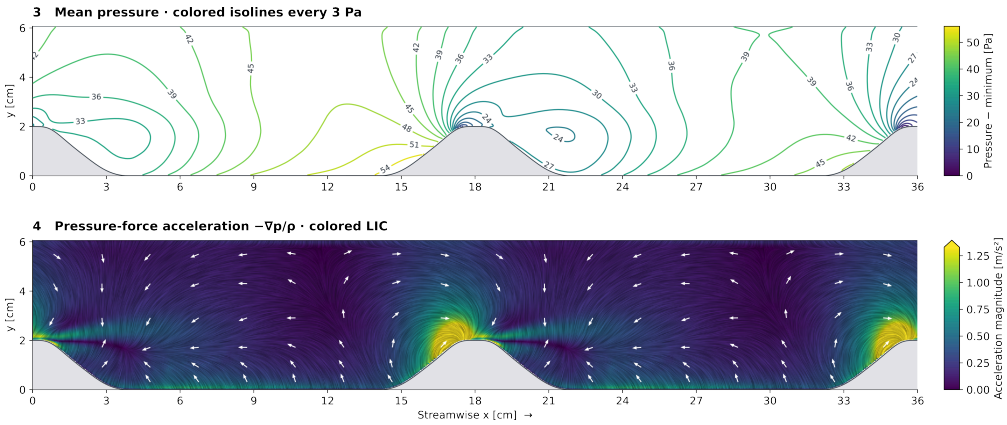


Figure 3.10: The mean pressure (top) and the pressure-force contribution per unit mass (bottom) in the flow from Figure 3.4 and Figure 3.7. The data is time-averaged, and displayed in a flow-wise plane in the middle of the channel.

The pressure diagram shows exactly the same information as was already shown in Figure 3.7, just plotted with isolines instead of with a colored field.

The pressure gradient diagram shows $-\vec{\nabla}p/\rho$. Like the velocity field from Figure 3.7, it is a vector field, represented with LIC and vector arrows.

Diagram by OC, built with GPT-6 Astra, based on source data kindly provided by Sylvain Laizet and Xuhui Zhou [8, 11]

Advice from an expert



Ponder for a moment what the dimensions of the physical properties in this equation 3/8 are. If you were to record in a file, to send it by email, the values for pressure in a given fluid domain, you would have to store one value (in Pa) for each point in space (x, y, z), once for each time point (t).

If you were to calculate the *effect* of pressure, you would calculate minus the gradient of pressure. To record this information, you would need to store *three* values (the three components, in x, y, and z, each in $\mathrm{Pa\,m^{-1}}$) for each point in space, once for each time point. This field of vectors would indicate the amount of force per unit volume with which pressure is pushing the fluid.

Having a good grasp of the tools we are using here is important, because things will soon get more complicated when viscous stress enters the room down below, ultimately leading to the all-powerful *Navier-Stokes equation* in the next chapter.

3.4 Special case: when we know pressure

There is a special class of problems where we can know the pressure easily: when the fluid is not moving (fluid statics).

3.4.1 Forces in static fluids

When the fluid is not moving and not accelerating, there are only two forces left acting on an infinitesimal cube:

$$\begin{aligned} \vec{F}_{\text{net, pressure}} + \vec{F}_{\text{gravity}} &= \vec{0} \\ -\,\mathrm{d}\mathscr{V}\,\vec{\nabla}p + \quad m\vec{g} &= \vec{0} \\ -\vec{\nabla}p + \quad \rho\vec{g} &= \vec{0} \end{aligned} \tag{3/9}$$

We can finally rewrite this as:

$$\vec{\nabla}p = \rho\vec{g} \tag{3/10}$$

in a static fluid.

This is a very useful equation, which states that in a static fluid, the main parameter affecting pressure is gravity. It reads as "the fluid density times the gravity vector is equal to the gradient (the change in space) of the pressure".

Advice from an expert



What this equation 3/10 is really saying is that in a static fluid, "pressure changes only with altitude". This sounds trivial, but consider the consequences, especially what is *not* in this equation. For example, when you swim under an anchored boat, you can't "feel" the presence of the boat: only your own depth matters. Or this: the pressure at the inlet of the turbines of a gigawatt-class hydraulic dam power station depends on the depth of the reservoir, but not at all on its overall size. So many consequences for such a small equation!

3.4.2 Pressure and depth

It is now possible to quantify pressure everywhere inside a static fluid. Most often when the fluid is static, the z-axis is oriented vertically, positive downwards. With this convention, there is no need for a vector equation to quantify pressure, and equation 3/10 becomes:

$$\frac{\mathrm{d}p}{\mathrm{d}z} = \rho g \tag{3/11}$$

in a static fluid, when z is oriented positive downwards.

The first consequence we draw from equation 3/11 is that in a static fluid (e.g. in a glass of water, in a swimming pool, in a calm atmosphere), pressure depends solely on height. In a static fluid, at a given depth, we will measure the same pressure regardless of the surroundings, as shown in Figure 3.11.

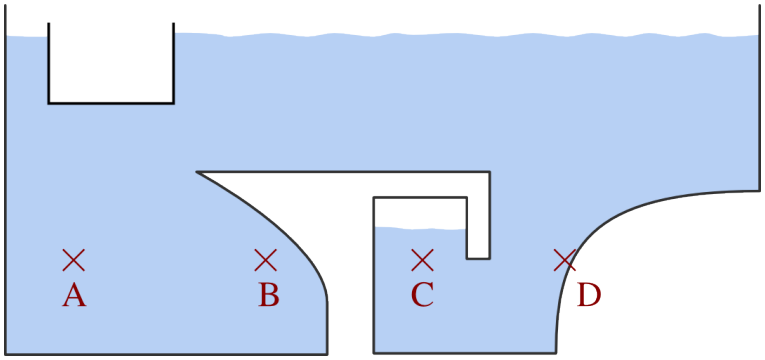


Figure 3.11: Pressure at a given depth (or height) in a static fluid does not depend on the environment. Here, as long as the fluid remains static, $p_A = p_B = p_C = p_D$.

Figure CC-0 OC



Video: when pressure-induced forces in static fluids matter: 24 hours of heavy tonnage transit through the *Miraflores* locks in Panama. Can you quantify the force acting on a single lock door?

by Y:Pancho507 (STYL)
https://youtu.be/LNKtS91jaxw

Let's put numbers into this equation. The density of liquid water is approximately constant: $\rho_{water} = 1000\ \mathrm{kg\,m^{-3}}$. In a water reservoir, equation 3/11 becomes:

$$\left(\frac{\mathrm{d}p}{\mathrm{d}z}\right)_{water} = \rho_{water}\, g$$
$$\left(\frac{\mathrm{d}p}{\mathrm{d}z}\right)_{water} = 1000 \times 9{,}81 = 9{,}81 \cdot 10^{3}\ \mathrm{Pa\,m^{-1}} = 9{,}81 \cdot 10^{-2}\ \mathrm{bar\,m^{-1}} \qquad (3/12)$$

So in static water, pressure increases by approximately $0{,}1\ \mathrm{bar\,m^{-1}}$ as depth increases. For example, at a depth of 3 m, the pressure will be approximately 1,3 bar (which is the atmospheric pressure plus $\Delta z \times \mathrm{d}p/\mathrm{d}z$).

In the atmosphere, the situation is more complicated, because the density ρ_{air} of atmospheric air is not uniform. If we model atmospheric air as a perfect gas, we can express the pressure gradient as:

$$\left(\frac{\mathrm{d}p}{\mathrm{d}z}\right)_{atm.} = \rho_{air}\, g = p\frac{1}{T}\frac{g}{R} \qquad (3/13)$$

This time, the variation of pressure with respect to distance depends on pressure itself (and it is proportional to it). At ambient conditions, we plug in 1 bar and 15 °C to obtain:

$$\left(\frac{\mathrm{d}p}{\mathrm{d}z}\right)_{atm.\ ambient} = 1 \cdot 10^{5} \times \frac{1}{288{,}15} \times \frac{9{,}81}{287}$$
$$= 11{,}86\ \mathrm{Pa\,m^{-1}} = 1{,}186 \cdot 10^{-4}\ \mathrm{bar\,m^{-1}} \qquad (3/14)$$

This is approximately 0,1 mbar/m, almost a thousand times smaller than that of water (Figure 3.12).

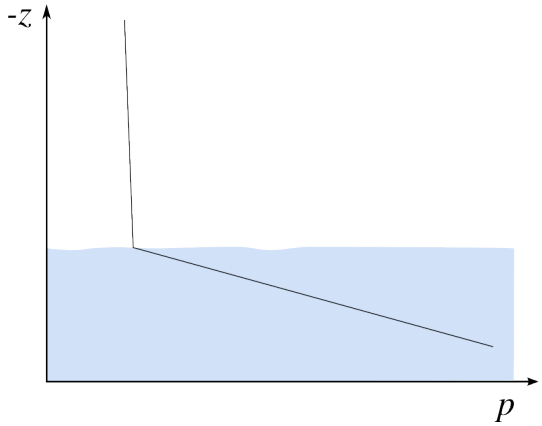
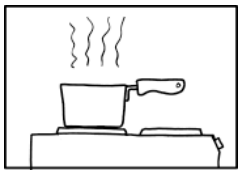


Figure 3.12: A quick diagram to represent pressure in the atmosphere and in water. The gradient of pressure with respect to altitude is almost a thousand times larger in water than in air.

Figure CC-0 OC

Since the rate of pressure change depends on pressure, it also varies with altitude, and the calculation of pressure differences in the atmosphere is a little more complicated than for water. When it is reasonable to consider that temperature T, the gravitational acceleration g and the gas constant R are the same everywhere, equation 3/13 can be integrated as follows:



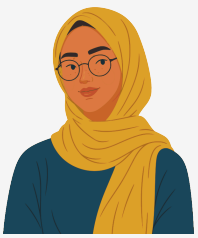
XKCD #2153: effects of high altitude
by Randall Munroe (CC-BY-NC)
https://xkcd.com/2153

$$\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}z} &= \frac{g}{RT_{\text{cst.}}}p \\ \int_1^2 \frac{1}{p}\,\mathrm{d}p &= \frac{g}{RT_{\text{cst.}}}\int_1^2 \mathrm{d}z \\ \ln\frac{p_2}{p_1} &= \frac{g}{RT_{\text{cst.}}}\Delta z \\ \frac{p_2}{p_1} &= \exp\left[\frac{g\Delta z}{RT_{\text{cst.}}}\right] \end{aligned} \qquad (3/15)$$

This is only a simplified model: in reality, air temperature varies significantly within the atmosphere, because air is mostly heated by the ground (the representative temperature gradient, called the tropospheric lapse rate, is $-6\,\mathrm{K\,km^{-1}}$). Adapting equation 3/15 for a uniform temperature gradient (instead of uniform temperature) is the subject of a problem sheet exercise.

Advice from an expert

Here, we see how the "p=$\rho g h$" equation that many of us learned in our first class of fluid mechanics is dangerous: if ρ is not uniform, everything falls apart. To be safe, when you study a static fluid, always start from equation 3/11. As you integrate p with respect to z, you will be forced to consider how ρ falls into the picture. In water, ρ is uniform (just a number). But in the atmosphere, air is "squished" by gravity, and much denser near the ground.



3.4.3 Buoyancy

There is nothing special about buoyancy. (Except for how the word sounds when you are trying to remain serious.) Any object immersed in a fluid is subjected to pressure on its walls. When the pressure is not uniform, then the net force due to fluid pressure on the body walls may be non-zero.



Video: playing around with an air pump and a vacuum chamber
by Y:Roobert33 (STYL)
https://youtu.be/ViuQKqUQ1U8

When the fluid is static, this net pressure force is called *buoyancy*. Then, the net pressure force is oriented upwards. The buoyancy force is completely independent of (and may or may not balance) the object's weight.

We saw with equation 3/10 that the variation of pressure in a static fluid is due only to the fluid's weight. So now, we can see that the force exerted on an immersed body is equal and opposite to the weight of the fluid it replaces (that is to say, the weight of the fluid that would occupy its own volume were it not there). This relationship is sometimes named *Archimedes' principle*. The force which results from the static pressure gradient applies to all immersed bodies: a submarine in an ocean, an object in a pressurized container, and of course, the reader of this document as presently immersed in the Earth's atmosphere.

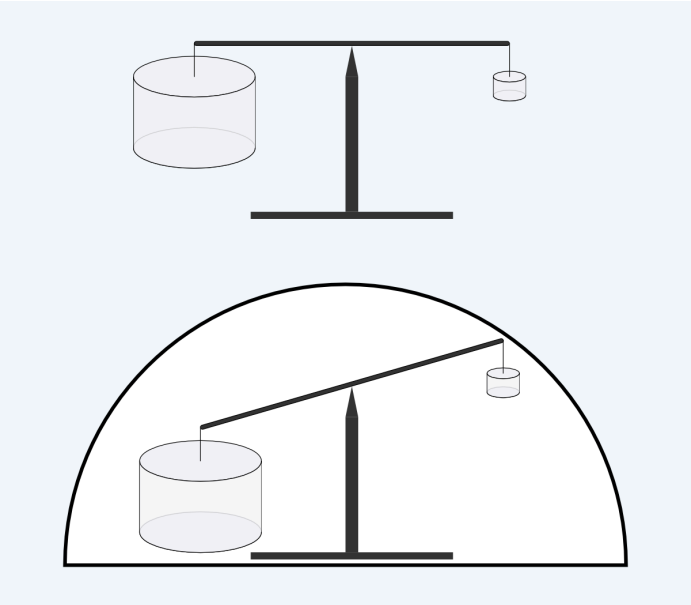


Figure 3.13: Immersion in a static fluid results in forces that depend on the body's volume. They can be demonstrated by the removal of the fluid (for example in a depressurized semi-spherical vessel).

Figure CC-0 OC

3.5 And what about viscous forces?

This will be added here next week!