

CHAPTER 2

Analysis of Existing Flows

✧ Quantify the forces and powers associated with an existing flow

2.1 What are we doing here?	2
2.2 A bit of accounting	2
2.3 Balance of mass	3
2.4 Balance of momentum	5
2.5 Balance of angular momentum	7
2.6 Balance of energy	8
2.7 The Bernoulli equation	11
2.8 Non-uniform flows and unsteady flows . . .	13
2.9 Limits of integral analysis	17

last modified: 2026-10-28

2.1 What are we doing here?

We want to figure out what is happening in flows that we can observe. In particular, we are interested in:

- accounting for mass (where is it going?)
- accounting for forces (why is the fluid turning or accelerating?)
- accounting for energy (who is paying for the fluid's transformation?).

For this, we will need to build some rigorous accounting system, and then we will write *balance equations*, which relate changes with one another.

2.2 A bit of accounting

In order to write useful equations, we need to begin with rigorous definitions, with the help of Figure 2.1:

- We call *control volume* a certain volume we are interested in. Fluid flows through the control volume. The control volume can change with time.
- At a certain moment, the mass of fluid that is inside the control volume is called the *system*. The system is traveling. At a later point in time, it has moved and deformed.

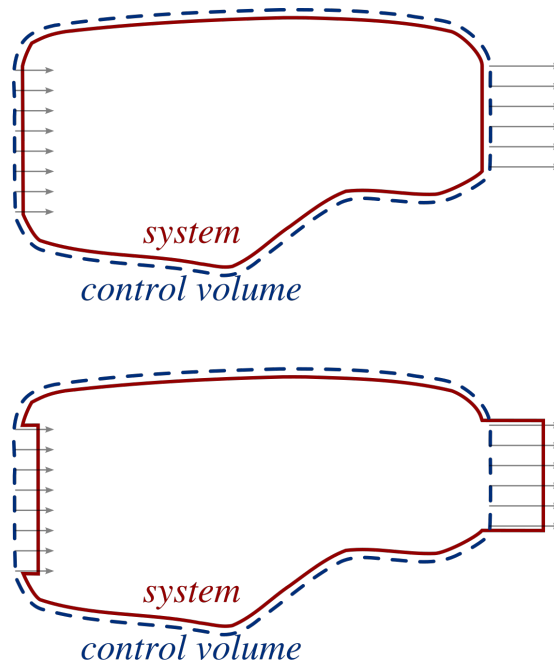


Figure 2.1: A control volume within a flow. The *system* is the amount of mass included within the control volume at a given time. Because mass enters and leaves the control volume, the system is being moved and deformed (bottom).

Figure CC-0 OC

We will start by considering cases where the fluid enters and leaves the control volume at a clearly-identifiable inlet and outlet, with uniform properties (only one velocity, one temperature etc.). This makes it easier to picture things in our minds. Later in section Section 2.8, we will look at more general cases. The method we develop here is called *integral analysis*, because it involves calculating the overall (integral) effect of the fluid flowing through a considered volume.

The four equations that we write in this chapter state that basic physical laws apply to the system. They are *balance equations*. Each time, we will *relate*

what is happening to the system to what we can measure at the inlet and outlet of the control volume. Let's dive in!

2.3 Balance of mass

2.3.1 The equation

How much mass flow ought to be coming in and out of the control volume? We answer this question by writing a mass balance equation. The mass m_{sys} of the system isn't changing. So logically, either mass is accumulating inside the control volume, or, it is flowing in and out of the sides:

$$\frac{dm_{\text{sys}}}{dt} = 0 = \frac{d}{dt}m_{\text{CV}} + \dot{m}_{\text{net}} \quad (2/1)$$

$\nearrow$ the rate of creation mass (zero) $\nearrow$ the storage rate inside the considered volume $\nearrow$ the net mass flow at the borders of the considered volume

We can say that the net mass flow $\dot{m}_{\text{net}}$ is a sum of two contributions, incoming and outgoing, and so our equation becomes:

$$0 = \frac{d}{dt}m_{\text{CV}} + \sum_{\text{incoming}} \dot{m} + \sum_{\text{outgoing}} \dot{m} \quad (2/2)$$

$\nearrow$ the rate of creation mass (zero) $\nearrow$ the storage rate inside the considered volume $\nearrow$ the sum of incoming mass flows (negative) $\nearrow$ the sum of outgoing mass flows (positive)

The sign convention is counter-intuitive: mass flows are negative inwards and positive outwards. This is vexing if you are used to counting flows in thermodynamics, but it will make sense once we account for momentum flows in the next section.

We can substitute $\dot{m} = \rho V_{\perp} A$ from chapter 1 into the equations above, obtaining:

$$0 = \frac{d}{dt}m_{\text{CV}} + \sum_{\text{incoming}} [\rho V_{\perp} A] + \sum_{\text{outgoing}} [\rho V_{\perp} A] \quad (2/3)$$

where $V_{\perp} = \vec{V} \cdot \vec{i}$ is positive inwards, and negative outwards.

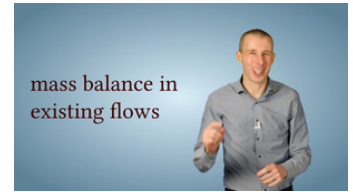
You can convince yourself that this is just an elaborate version of our "good old" mass flow equation 1/4 if you consider the case where the flow is completely steady ($d/dt = 0$) and has only one inlet and one outlet:

$$0 = 0 + [\rho V_{\perp} A]_{\text{inlet}} + [\rho V_{\perp} A]_{\text{outlet}}$$

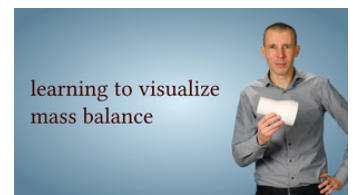
$$(\rho |V_{\perp}| A)_1 = (\rho |V_{\perp}| A)_2 \quad (2/4)$$

when the flow is steady,
through a fixed control volume,
with only one inlet and one outlet.

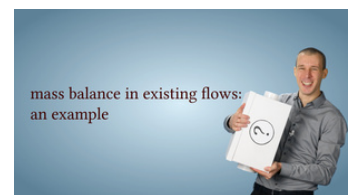
The mass balance equation is often called *continuity equation* in the literature.



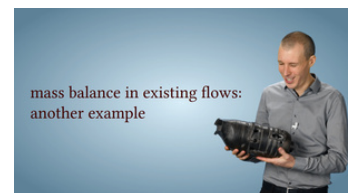
Learnfluidmechanics.com video 2.1:
mass balance equation
by Olivier Cleynen
<https://learnfluidmechanics.com/>



Learnfluidmechanics.com video 2.2:
Visualizing mass balance
by Olivier Cleynen
<https://learnfluidmechanics.com/>



Learnfluidmechanics.com video 2.3:
Mass balance: an example
by Olivier Cleynen
<https://learnfluidmechanics.com/>



Learnfluidmechanics.com video 2.4:
Mass balance: another example
by Olivier Cleynen
<https://learnfluidmechanics.com/>

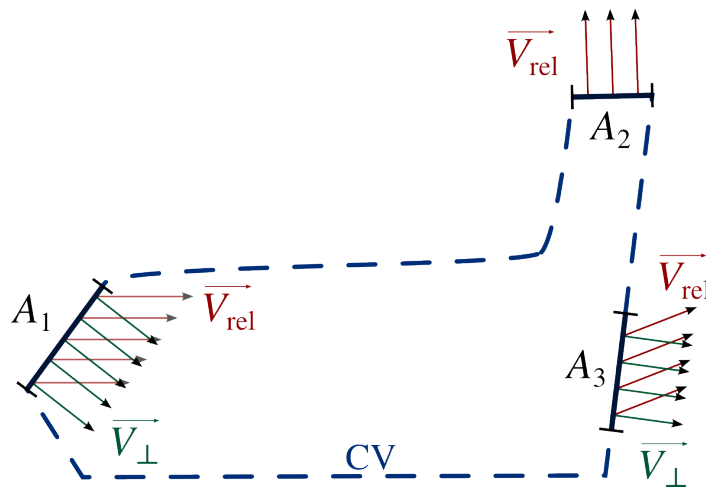


Figure 2.2: A control volume for which the system's properties are uniform at each inlet and outlet. Here Equation (2/3) translates as

$$0 = \frac{d}{dt} m_{CV} - \rho_1 |V_{11}| A_1 + \rho_2 |V_{12}| A_2 + \rho_3 |V_{13}| A_3.$$

Figure CC-0 OC

2.3.2 Paradox: obstructions make it go faster

We need to engage right away with one paradox embedded in Equation (2/4). Let's take a look at the outlet, where the mass flow is the product $\rho V_{\perp} A$. We can think of many different variations of these three parameters, as in Figure 2.3:

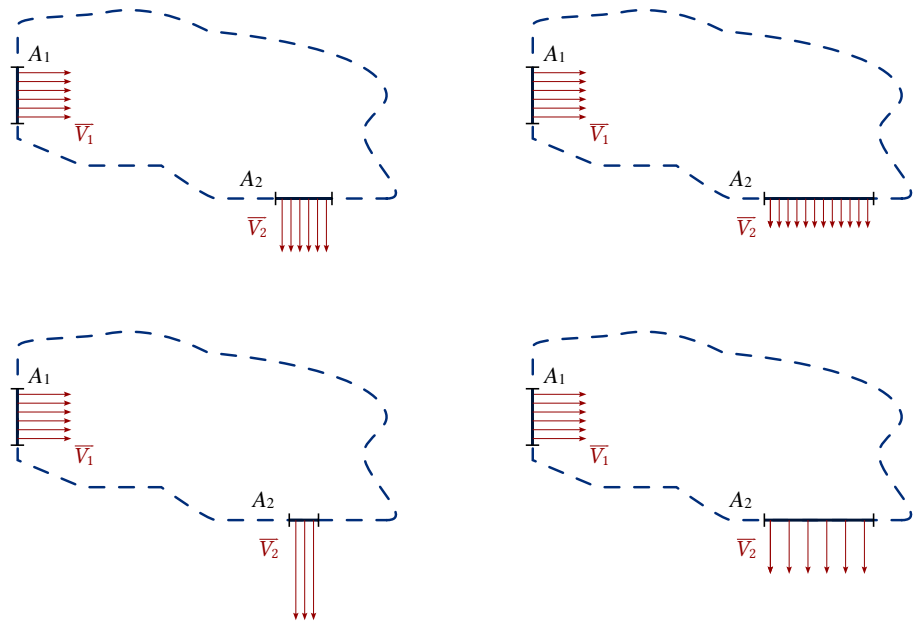


Figure 2.3: In these four control volumes, the outlet mass flow is the same. We can change the outlet area and even the density. Is reducing the outlet area a way to get high outlet velocity, for free?

Figure CC-0 OC

The obvious (wrong) conclusion here is that “if A decreases, then V must increase”. Surely something must prevent us from tending towards infinite velocity simply by “squeezing” the outlet down toward zero cross-section area?

Advice from an expert

As we climb into ever more complex mathematical descriptions of the dynamics of fluids, you must relentlessly confront the paradoxes that you find on the way. Don't become one of these aces who manipulate the math all day, but then can't figure out basic physical trends. When you squeeze the outlet of the water hose in the garden, water shoots out faster. What is the limit though? Just saying "unaccounted losses" does not cut it — note that Equation (2/1) says *nothing* about anything getting lost or neglected. Try to formulate an answer before you read on!



The answer to the paradox is that four things can happen when we reduce the outlet cross-section area:

1. The outlet velocity $V_{\perp}$ may increase, as we first expect.
2. Mass may accumulate inside the control volume. This is the term $d/dt(m_{CV})$ in Equation (2/1).
3. The density ρ may increase. In low-speed flows without heat transfer, ρ does not vary significantly (see chapter 1). But in compressible flows (for example when combustion is involved, or when compressed air is expanded), ρ may vary together with A and V , sometimes in very surprising ways.
4. The mass flow $\dot{m}$ itself may change. There is no causal relationship in equation Equation (2/4). In an incompressible flow, it may well be that reducing A_2 leads to an increase in V_2 , but nothing guarantees that the product of the two remains constant. In other words, reducing A_2 may both increase V_2 and decrease $\dot{m}$. Increases in velocity are not "for free": they require force be applied and energy be spent. The mass balance equation cannot account for those phenomena.

Advice from an expert

Remember the title of the chapter: the tools here are for analyzing *existing* flows: those for which we can, if needed, gather more information by making measurements. If you find yourself *predicting* velocity in a machine you design with just a mass balance equation, then you might quickly find yourself making unrealistic assumptions. Immediately check what force and power are required to generate this velocity. For this, you need a momentum balance equation, and an energy balance equation.

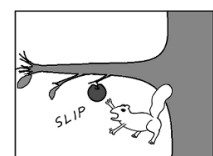


2.4 Balance of momentum

2.4.1 The equation

What *force* applies to the fluid as it travels through our control volume? We'll answer this question by writing a momentum balance equation. Let's start by reformulating Newton's second law, the famous $\vec{F}_{\text{net}} = m\vec{a}$, into

$$\vec{F}_{\text{net}} = \frac{d}{dt}(m\vec{V}) \quad (2/5)$$



Abstruse Goose #338: Newton's laws of motion almost didn't happen
by an anonymous artist (CC-BY-NC)
<https://abstrusegoose.com/338>

where $\vec{F}_{\text{net}}$ is the net sum of forces applying to an object (N),
and $m\vec{V}$ is that object's momentum (kg m s^{-1}).

Take some time to think about this: we are calculating the rate of change of a *vector* now. Changing a vector may mean that it gets larger or smaller (the object's speed is increasing/decreasing), but it can also mean the vector is changing direction (the object is changing direction).

And now, let's think of our system inside our control volume. The net force applying on the system is the time change of its momentum. But where is this momentum going? We answer this with a *momentum balance* equation:

$$\frac{d}{dt}(m\vec{V}_{\text{sys}}) = \vec{F}_{\text{net}} = \frac{d}{dt}(m\vec{V})_{\text{cv}} + (\dot{m}\vec{V})_{\text{net}} \quad (2/6)$$

$\uparrow$ the net force on the fluid $\uparrow$ the rate of change of momentum in the considered volume $\uparrow$ the net momentum flow at the borders of the considered volume

Just like we did with the mass equation, we can split the “net” flow term into two sums, one for incoming and one for outgoing flows:

$$\vec{F}_{\text{net}} = \frac{d}{dt}(m\vec{V})_{\text{cv}} + \sum_{\text{incoming}} (\dot{m}\vec{V}) + \sum_{\text{outgoing}} (\dot{m}\vec{V}) \quad (2/7)$$

$\uparrow$ the net force (vector sum of forces) on the fluid $\uparrow$ the rate of change of momentum in the CV (“slosh”) $\uparrow$ the incoming momentum flow ($\dot{m}$ negative) $\uparrow$ the outgoing momentum flow ($\dot{m}$ positive)

The same convention as above is applied for the sign of the mass flow $\dot{m}$. For example, in a case where there were one inlet and one outlet, we would write:

$$\begin{aligned} \vec{F}_{\text{net}} &= \frac{d}{dt}(m\vec{V})_{\text{cv}} + (\dot{m}\vec{V})_{\text{in}} + (\dot{m}\vec{V})_{\text{out}} \\ &= \frac{d}{dt}(m\vec{V})_{\text{cv}} - (|\dot{m}| \vec{V})_{\text{in}} + (|\dot{m}| \vec{V})_{\text{out}} \end{aligned}$$

when there is only one inlet and one outlet.

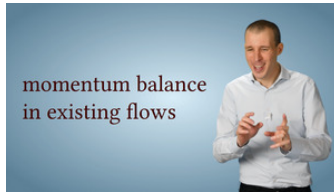
As before, we can substitute $\dot{m} = \rho V_{\perp} A$ into the equations above, obtaining:

$$\vec{F}_{\text{net on fluid}} = \frac{d}{dt}(m\vec{V})_{\text{cv}} + \sum_{\text{incoming}} [\rho V_{\perp} A \vec{V}] + \sum_{\text{outgoing}} [\rho V_{\perp} A \vec{V}] \quad (2/8)$$

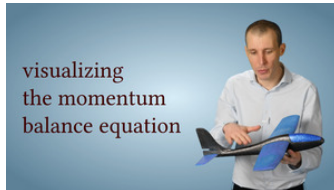
For steady flow through a fixed considered volume,
where $V_{\perp}$ is negative inwards, positive outwards.

In the example case where there is one inlet and one outlet, we would write:

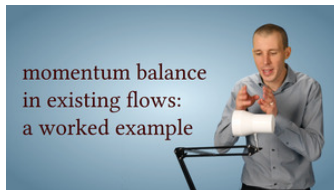
$$\begin{aligned} \vec{F}_{\text{net}} &= \frac{d}{dt}(m\vec{V})_{\text{cv}} + (\rho V_{\perp} A \vec{V})_{\text{in}} + (\rho V_{\perp} A \vec{V})_{\text{out}} \\ &= \frac{d}{dt}(m\vec{V})_{\text{cv}} - (\rho |V_{\perp}| A \vec{V})_{\text{in}} + (\rho |V_{\perp}| A \vec{V})_{\text{out}} \end{aligned}$$



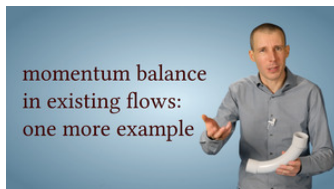
Learnfluidmechanics.com video 2.5:
Momentum balance
by Olivier Cleynen
<https://learnfluidmechanics.com/>



Learnfluidmechanics.com video 2.6:
Visualizing momentum balance
by Olivier Cleynen
<https://learnfluidmechanics.com/>



Learnfluidmechanics.com video 2.7:
Momentum balance: an example
by Olivier Cleynen
<https://learnfluidmechanics.com/>



Learnfluidmechanics.com video 2.8:
Momentum balance: another example
by Olivier Cleynen
<https://learnfluidmechanics.com/>

2.4.2 A bit of exploration

To explore what this equation can tell us, let's focus on the simple case where the control volume sees by a steady flow (d/dt) with mass flow $\dot{m}$, with one inlet (point 1) and one outlet (point 2). The net force $\vec{F}_{\text{net}}$ applying on the fluid is

$$\vec{F}_{\text{net}} = |\dot{m}| (\vec{V}_2 - \vec{V}_1) \quad (2/9)$$

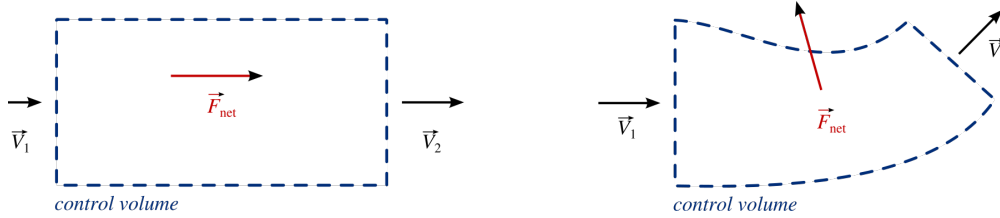


Figure 2.4: Two reasons can explain why a net force $\vec{F}_{\text{net}}$ appears in eq. Equation (2/9). On the left, $\vec{V}_2$ and $\vec{V}_1$ are aligned, but have different lengths.

On the right $\vec{V}_2$ and $\vec{V}_1$ have the same length, but different directions.

Figure CC-0 OC

Three remarks can be made about this equation. First, we need to be aware that this is not one, but *three* equations, one for each dimension. In order to express $\vec{F}_{\text{net}}$, we need to calculate its three components:

$$\begin{aligned} F_{\text{net } x} &= |\dot{m}| (V_{2x} - V_{1x}) \\ F_{\text{net } y} &= |\dot{m}| (V_{2y} - V_{1y}) \\ F_{\text{net } z} &= |\dot{m}| (V_{2z} - V_{1z}) \end{aligned} \quad (2/10)$$

Second, there are *two* reasons why we could calculate a non-zero net force on the fluid in Equation (2/9), as illustrated in Figure 2.4.

1. Even if $\vec{V}_2$ is aligned and in the same direction as $\vec{V}_1$, they can be of different magnitude. A force is required to accelerate or decelerate the fluid (more precisely, an acceleration or deceleration of the fluid is equivalent to a force);
2. Even if $\vec{V}_2$ has the same magnitude as $\vec{V}_1$, they can have different directions. A force is required to change the direction in which a flow is flowing (or more precisely, a change of direction is equivalent to a force).

The final remark is that the equation does not describe a cause-effect relationship. The net force does not cause the change in velocity any more than the change in velocity causes the net force: they are both equivalent and simultaneous. Similarly, we have no way to know what $\vec{F}_{\text{net}}$ is made of. The exact mechanism which adds up to a net force (pressure change, shear applied through a static wall, the movement of a turbine, etc.) is “hidden” in the control volume, and unknown to us. In order to find out what happens inside the control volume, we need a different set of tools, that will arrive in chapter 4.

2.5 Balance of angular momentum

What moment $\vec{M}$ (“twisting effort”) is being applied to the fluid as it travels through our control volume? We answer this question by writing an angular momentum balance.



Video: rocket landing gone wrong. Can you compute the moment exerted by the top thruster around the base of the rocket as it (unsuccessfully) attempts to compensate for the collapsed landing leg?

by Y:SciNews (STVL)
<https://youtu.be/4cvGGxTsQx0>



Abstruse Goose #237: how many dimensions are enough for you?
 by an anonymous artist (CC-BY-NC)
<https://abstrusegoose.com/237>

We position ourselves at a point X , about which we measure all moments. All positions are measured with a position vector $\vec{r}_X$. The angular momentum $\vec{H}$ of something about point X is the cross-product $\vec{r}_X \times m\vec{V}$. The vector sum of all moments applying to it is the rate of change of $\vec{H}$:

$$\vec{M}_{\text{net}} = \frac{d\vec{H}}{dt} \quad (2/11)$$

Now, when we apply this physical law to the system transiting in our control volume, we get:

$$\frac{d\vec{H}_{\text{sys}}}{dt} = \vec{M}_{\text{net}} = \frac{d}{dt}\vec{H}_{\text{CV}} + (\vec{r}_X \times m\vec{V})_{\text{net}} \quad (2/12)$$

We can reformulate this by splitting the net angular momentum flow into two parts:

$$\vec{M}_{\text{net}} = \frac{d}{dt}\vec{H}_{\text{CV}} + \sum_{\text{incoming}} (\vec{r}_X \times m\vec{V}) + \sum_{\text{outgoing}} (\vec{r}_X \times m\vec{V}) \quad (2/13)$$

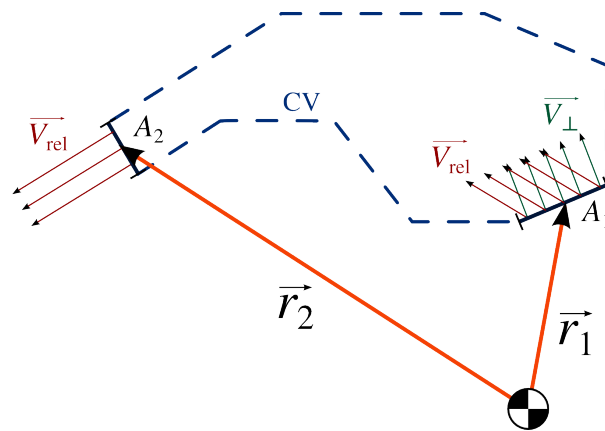


Figure 2.5: A control volume with a reference point, on the bottom right, from which to calculate angular momentum flows, and moments.

Figure CC-0 OC

Advice from an expert

The angular momentum balance equation is useful in cases where we attempt to balance a machine using fluid flows. Moments can be added and subtracted as vectors, just like forces. It's an extra layer of abstraction to learn (just about every fluid dynamicist has suffered at first with the clockwise-is-positive convention). You'll be glad you learned it when you attempt to prevent the helicopter from spinning uncontrollably – or when you ask your CFD software to calculate a moment, and it spits out a vector!



2.6 Balance of energy

2.6.1 The equation

What power is involved in the fluid traveling through the control volume? We answer this question by writing an energy balance equation.

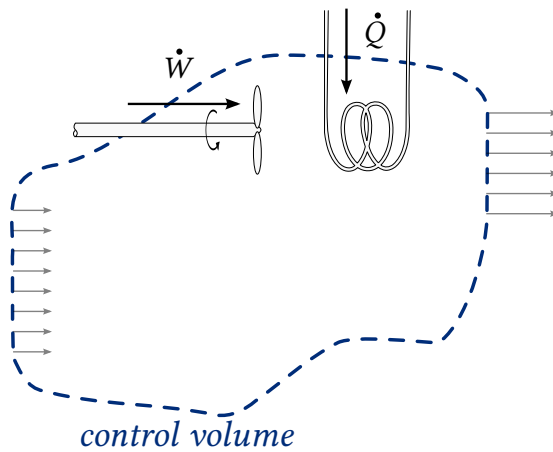


Figure 2.6: A control volume with two power transfers, in which we can write a balance of energy.

Figure CC-0 OC

Here we prefer to express the energy E as the specific energy e (in J kg^{-1}) multiplied by the mass m (kg). The time rate change of me is measured in watts ($1 \text{ W} \equiv 1 \text{ J s}^{-1}$). The energy balance equation is then:

$$\frac{d}{dt}(E_{\text{sys}}) = \dot{Q} + \dot{W} = \frac{d}{dt}(me)_{\text{cv}} + (\dot{me})_{\text{net}} \quad (2/14)$$

$\uparrow$
 the net power
transferred into
the considered
volume

$\uparrow$
 the rate of
accumulation
of energy in
the considered
volume

$\uparrow$
 the net energy
flow at the
borders of the
considered
volume

where $\dot{Q}$ is the net power transferred as heat ($\dot{W}$, positive inwards)
and $\dot{W}$ the power transferred as work ($\dot{W}$, positive inwards)

You may be familiar with this description if you have already studied engineering thermodynamics:

- the net power transferred as heat is $\dot{Q}_{\text{net}}$ (positive inwards);
- the net power transferred as work is $\dot{W}_{\text{surfaces, net}}$ (for example, a moving piston, turbine blade, or rotating shaft, positive inwards).

We can reformulate the last term as the sum of incoming and outgoing energy flows:

$$\dot{Q} + \dot{W} = \frac{d}{dt}(E)_{\text{cv}} + \sum_{\text{incoming}} (\dot{me}) + \sum_{\text{outgoing}} (\dot{me}) \quad (2/15)$$

$\uparrow$
 the net power
transferred into
the considered
volume
(positive
inwards)

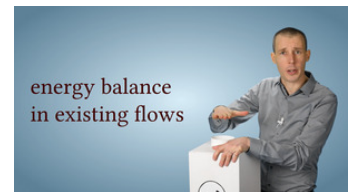
$\uparrow$
 the rate of
accumulation
of energy in
the considered
volume

$\uparrow$
 the incoming
energy flow ($\dot{m}$
negative)

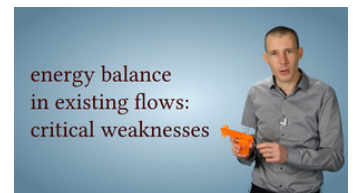
$\uparrow$
 the outgoing
energy flow ($\dot{m}$
positive)

Turning now to the specific energy e , we break it down into components:

- the specific internal energy i , which represents the energy per unit mass contained as stored heat within the fluid itself. In thermodynamics, this is often noted u , but in fluid mechanics we reserve this symbol to note the x -component of velocity. In a perfect gas, i is simply proportional to absolute temperature ($i = c_v T$), but for other fluids such as water, it cannot be easily



Learnfluidmechanics.com video 2.9:
Energy balance
by Olivier Cleynen
<https://learnfluidmechanics.com/>



Learnfluidmechanics.com video 2.10:
Energy balance: shortcomings
by Olivier Cleynen
<https://learnfluidmechanics.com/>

measured, and precomputed tables relating i to other properties must be used;

- the specific power that is provided or received by the fluid in order to enter and leave the considered volume. This is called *power to cross a surface* or insertion/extraction power; for each inlet or outlet we have

$$w_{\text{cross-surface}} = \frac{p}{\rho} \quad (2/16)$$

- the specific kinetic energy e_k ,

$$e_k \equiv \frac{1}{2} V^2 \quad (2/17)$$

- the specific potential energy e_p , related to gravity g (m s^{-2}) and altitude z (m) as:

$$e_p \equiv gz \quad (2/18)$$

We can now rewrite the “energy” e as a sum of terms:

$$(\dot{m} e)_{\text{net}} = \left[\dot{m} \left(i + \frac{p}{\rho} + \frac{1}{2} V^2 + gz \right) \right]_{\text{net}} \quad (2/19)$$

And our energy balance equation becomes:

$$\dot{Q} + \dot{W} = \frac{d}{dt}(E)_{\text{cv}} + \sum_{\text{incoming}} \left[\dot{m} \left(i + \frac{p}{\rho} + \frac{1}{2} V^2 + gz \right) \right] + \sum_{\text{outgoing}} \left[\dot{m} \left(i + \frac{p}{\rho} + \frac{1}{2} V^2 + gz \right) \right] \quad (2/20)$$



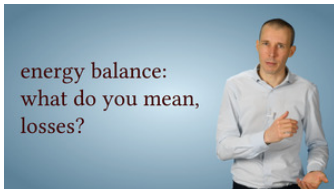
Learnfluidmechanics.com video 2.11:
Energy balance: an example
by Olivier Cleynen
<https://learnfluidmechanics.com/>

In case there is only one inlet and one outlet, we have just “one” mass flow $\dot{m}$ and the mass balance equation can be re-expressed

$$\dot{Q} + \dot{W} = \frac{d}{dt}(E)_{\text{cv}} + |\dot{m}| \left[\Delta i + \Delta \frac{p}{\rho} + \Delta \left(\frac{1}{2} V^2 \right) + \Delta(gz) \right] \quad (2/21)$$

For steady flow through a fixed considered volume,
where $\dot{m} = \rho V_{\perp} A$ is negative inwards, positive outwards.

This last equation is often called *steady flow energy equation* in thermodynamics (where it is usually expressed with the help of the concept of *enthalpy* $h \equiv i + p/\rho$, which we do not use here).



Learnfluidmechanics.com video 2.12:
Energy balance: “losses”
by Olivier Cleynen
<https://learnfluidmechanics.com/>

2.6.2 Critical weaknesses

This equation is very useful in principle, but not so much in practice, for two reasons. First, it contains a lot of terms. There are five fluid properties at inlet, five at outlet which affect energy, and it is difficult to predict which one will be affected by a heat or work transfer. For example, consider a simple water pump with known powers $\dot{Q}$ and $\dot{W}$. An efficient pump will generate large increases in p (or V and z), while an inefficient pump will generate large increases in i and $1/\rho$. The energy balance equation, in this form, tells us nothing about how the heat and work entering the control volume are redistributed among the twelve (!) remaining terms.

Advice from an expert

Again, remember the title of the chapter. To calculate the value of any one property in equation Equation (2/21), you need to input the value of the *thirteen* other ones. It is tempting to take shortcuts while doing so (“oh, the pressure is probably the same”), with disastrous consequences. There is no solution to this. If you are attempting to predict fluid flow, and are missing information, better stop without a result than take hazardous attempts at using equation Equation (2/21).



A second critical weakness in this equation is that the terms have disproportionate values in practice. The heat capacity of ordinary fluids is very large, and so i is usually hundreds of times larger than the four terms in the brackets of equation Equation (2/21). In water for example, an increase of temperature of $0,1\text{ }^{\circ}\text{C}$ (with the term Δi) requires the same energy as increasing its velocity from 30 km h^{-1} to 110 km h^{-1} (with the term Δe_k). This is not an issue in thermodynamics, where heat, work and temperature are the most important parameters. But in fluid mechanics, velocity is of great interest, and the energy balance is not always useful to predict its changes.

Advice from an expert

In fluid dynamics, fluid movement usually involves relatively small amounts of energy. You can convince yourself of this by dropping milk into a bowl of water: minuscule amounts of potential energy as $\Delta(gz)$ are converted into an incredibly complex distribution of velocities, before slowly dissipating into internal energy as Δi .

While *in principle*, we could calculate pressure drops or velocity changes by measuring temperature differences (and thus Δi), in practice this only works when very high powers are involved, such as in a compressor or in a rocket engine nozzle. For ordinary flow (say, air flow around a car, or water flow in a pipe), the temperature changes are much too small to be measured.



2.7 The Bernoulli equation

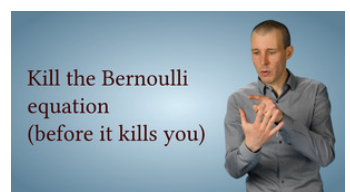
2.7.1 How it works

The Bernoulli equation is the energy equation with added restrictions.

To write the Bernoulli equation, we will start from equation Equation (2/21) and add five constraints:

1. Steady flow. So $d/dt(me)_{cv}$ will be zero;
2. Incompressible flow. Thus, ρ stays constant;
3. No heat or work transfer. Thus, $\dot{Q}_{net}$ and $\dot{W}_{shaft, net}$ are zero;
4. No friction. So, the fluid internal energy i cannot increase;
5. One-dimensional flow. Thus, our considered volume has only one inlet (labeled 1) and one outlet (labeled 2): all fluid particles move together with the same transit time, and the overall trajectory is already known.

With these five restrictions, equation Equation (2/21) simply becomes:



Learnfluidmechanics.com video 2.13:
Kill the Bernoulli equation
by Olivier Cleynen
<https://learnfluidmechanics.com/>

$$0 + 0 = \left[\dot{m} \left(i_{\text{cst.}} + \frac{p}{\rho_{\text{cst.}}} + \frac{1}{2} V^2 + gz \right) \right]_1 + \left[\dot{m} \left(i_{\text{cst.}} + \frac{p}{\rho_{\text{cst.}}} + \frac{1}{2} V^2 + gz \right) \right]_2$$

Dividing by $|\dot{m}|$ and canceling $i_{\text{cst.}}$, as follows,

$$0 = - \left(i_{\text{cst.}} + \frac{p}{\rho_{\text{cst.}}} + \frac{1}{2} V^2 + gz \right)_1 + \left(i_{\text{cst.}} + \frac{p}{\rho_{\text{cst.}}} + \frac{1}{2} V^2 + gz \right)_2$$

$$0 = - \left(\frac{p}{\rho_{\text{cst.}}} + \frac{1}{2} V^2 + gz \right)_1 + \left(\frac{p}{\rho_{\text{cst.}}} + \frac{1}{2} V^2 + gz \right)_2$$

and multiplying by the (constant and uniform) density ρ , we obtain the **Bernoulli equation**, with all terms having dimensions of pressure:

$$\left(p + \frac{1}{2} \rho V^2 + \rho gz \right)_1 = \left(p + \frac{1}{2} \rho V^2 + \rho gz \right)_2 \quad (2/22)$$

This equation describes the properties of a fluid particle in a steady, incompressible, friction-less flow with no energy transfer.

2.7.2 The Bernoulli equation will kill you

Take a moment to ponder how incredibly frustrating the restrictions brought by the five conditions above:

1. Steady flow.

This constrains us to continuous flows with no transition effects, which is a reasonable limit;

2. Incompressible flow.

We cannot use this equation to describe flow in compressors, turbines, diffusers, nozzles, nor in flows where $M > 0.6$.

3. No heat or work transfer.

We cannot use this equation in a machine (e.g. in pumps, turbines, combustion chambers, coolers).

4. No friction.

This is a tragic restriction! We cannot use this equation to describe a turbulent or viscous flow, e.g. near a wall or in a wake.

5. One-dimensional flow.

This equation is only valid if we know precisely the trajectory of the fluid whose properties are being calculated.

Of course, we can overcome these shortcomings by adding one extra (negative) term called “ Δp_{loss} ” to eq. Equation (2/22), which lumps together all of the effects unaccounted for. In this way, we obtain the **Bernoulli equation with losses**:

$$\left(p + \frac{1}{2} \rho V^2 + \rho gz \right)_1 = \left(p + \frac{1}{2} \rho V^2 + \rho gz \right)_2 + \Delta p_{\text{loss}} \quad (2/23)$$

There are indeed cases where the pressure losses due to the imperfection of the flow are well-understood, and can be easily quantified. This is true of flow in pipes, for example (we study these later). In these cases, eq. Equation (2/23) is extremely useful.

However, this approach is also easily misused. In a fluid flow where the simplifications above don’t apply—and many such flows can be found in everyday life as well as engineering applications!—Equation (2/23) will betray you. Convince yourself that *any wrong equation* can be made correct by adding an unknown “bucket” term at the end: for example $2 + 3 = -18 + \Delta p_{\text{loss}}$.

Advice from an expert

In case you are not sure whether the Bernoulli equation applies, *start from an energy balance equation*. Crossing out the terms that do not apply will force you to question their importance (e.g. is heat transfer really negligible? etc.). Remember that Johan Bernoulli's work came 69 years before Young coined the term “energy”, and a whole 163 years before the Wright Brothers' first flight.

If you do not come to a conclusive end, do not remove terms that are inconvenient. The unfortunate reality is that in fluid mechanics, the energy balance equation contains many terms, with disproportionate values, and using it alone is not enough to solve most practical problems.

*Advice from an expert*

Among the five restrictions listed, the last is the most severe, and the most often forgotten: *the Bernoulli equation does not allow us to predict the trajectory of fluid particles*. Just like all of the other equations in this chapter, it requires a control volume with a known inlet and a known outlet. If you find yourself drawing out flow streamlines and interpreting the result with the Bernoulli equation, you are running astray. The tools you need to do this correctly are waiting for us in chapter 4.



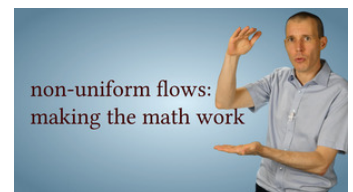
2.8 Non-uniform flows and unsteady flows

2.8.1 The idea

Not all flows have just one inlet and one outlet. We want to quantify things in any arbitrary flow, with any control volume, as shown for example in Figure 2.7. This will lead us to re-write our balance equations, each in a more capable, general (and complicated!) way.



Learnfluidmechanics.com video 2.14:
Non-uniform flows
by Olivier Cleynen
<https://learnfluidmechanics.com/>



Learnfluidmechanics.com video 2.15:
Non-uniform flows: basic example
by Olivier Cleynen
<https://learnfluidmechanics.com/>

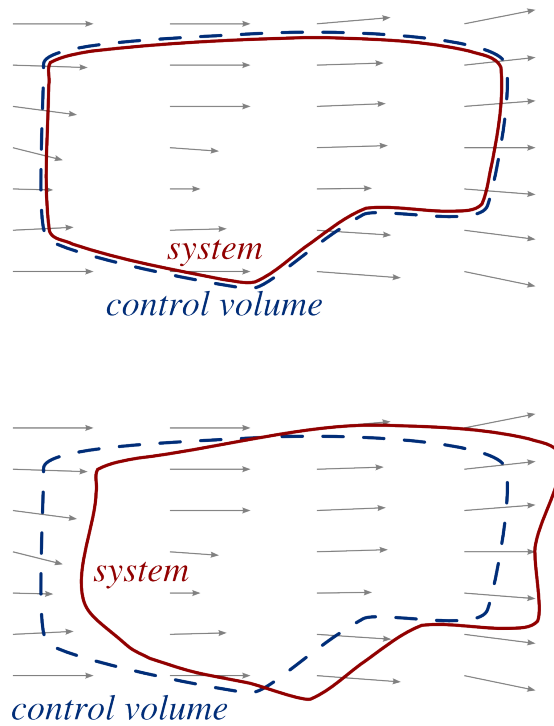


Figure 2.7: A control volume within an arbitrary flow. The control volume may also change shape with time, although this is not represented here.

Figure CC-0 OC

2.8.2 More powerful accounting

We are going to proceed a bit differently now: we will first prepare the flow accounting rigorously, and then the implementation will be just a matter of replacing variables.

We start with a “dummy” fluid property B , an arbitrary additive property of the fluid. By the term *additive* property, we mean that the total *amount of property* is divided if the fluid is divided. For instance, this works for properties such as mass, volume, energy, entropy, but not pressure or temperature.

The *specific* (i.e. per unit mass) value of B is designated $b \equiv B/m$.

We now want to compute the variation of a system’s property B based on measurements made at the borders of the control volume. Our basic accounting equation has three parts, which we are familiar with:

- The rate of change of the quantity B of the system is measured with the term dB_{sys}/dt
- The control volume can accumulate some of the “ B ”, and so its amount will vary as dB_{CV}/dt .
- The flux of B through the boundaries is going to be $\dot{B}_{\text{net}}$.

We can now link these three terms with the simple equation:

$$\frac{dB_{\text{sys}}}{dt} = \frac{dB_{\text{CV}}}{dt} + \dot{B}_{\text{net}} \quad (2/24)$$

Now come two modifications. First B may not be uniformly distributed within the control volume, and so we like to express the term B_{CV} as the integral of the volume density $B/\mathcal{V}$ with respect to volume $\mathcal{V}$:



Learnfluidmechanics.com video 2.16:
Unsteady effects

by Olivier Cleynen
<https://learnfluidmechanics.com/>

$$B_{CV} = \iiint_{CV} \frac{B}{\mathcal{V}} d\mathcal{V} = \iiint_{CV} \rho b d\mathcal{V} \quad (2/25)$$

$$\frac{dB_{CV}}{dt} = \frac{d}{dt} \iiint_{CV} \rho b d\mathcal{V}$$

And second, the flow of B needs to be measured with an integral, because the values may change across the surface. Now we choose to express $\dot{B}_{net}$ by quantifying, for each area element dA of the control volume's surface, the surface flow rate $\rho b V_{\perp}$ of property B that flows through it, as shown in Figure 2.8. The integral over the entire control volume surface CS of this term is:

$$\dot{B}_{net} = \iint_{CS} \rho b V_{\perp} dA = \iint_{CS} \rho b (\vec{V}_{rel} \cdot \vec{n}) dA \quad (2/26)$$

where CS is the control surface (enclosing the control volume CV),
 $\vec{n}$ is a unit vector on each surface element dA pointing outwards,
 $\vec{V}_{rel}$ is the local velocity of fluid relative to the control surface,
 and $V_{\perp} \equiv \vec{V}_{rel} \cdot \vec{n}$ is the local cross-surface speed (positive outwards, neg. inwards)

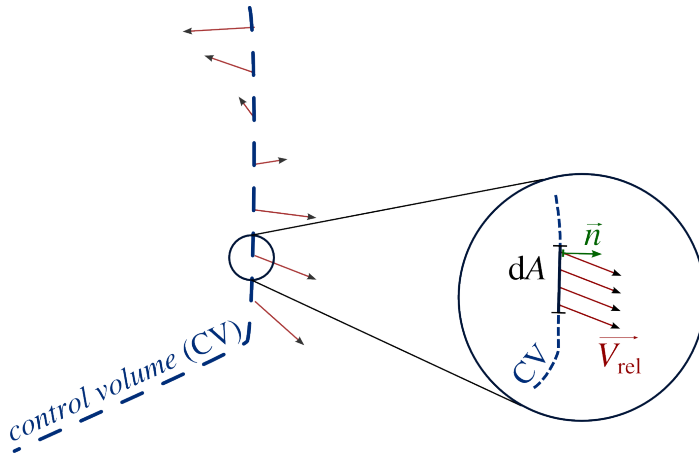


Figure 2.8: Part of the system may be flowing through an arbitrary piece of the control surface with area dA . The $\vec{n}$ vector defines the orientation of dA surface, and by convention is always pointed outwards. The amount of B flowing through this small area per unit time is dA is $\rho V_{\perp} dA b$

Figure CC-0 OC

By inserting Equation (2/25) and Equation (2/26) into equation Equation (2/24), we obtain:

$$\frac{dB_{sys}}{dt} = \frac{d}{dt} \iiint_{CV} \rho b d\mathcal{V} + \iint_{CS} \rho b (\vec{V}_{rel} \cdot \vec{n}) dA \quad (2/27)$$

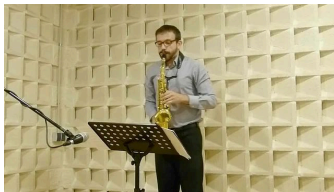
Equation (2/27) is named the *Reynolds transport theorem*^w. It is written for vaguely abstract property B , which we will now replace with physical properties to get useful new versions of our balance equations:

Advice from an expert

Take some time to observe the (sometimes curious!) terms of equation Equation (2/27), and learn the associated vocabulary. Just like an accountant in a company would like to have a very clear method for counting how money is spent and earned, fluid dynamicists need very good tools to describe what's coming in and out of their flows. When you'll be hurling three-dimensional vector operations at swirling flows later, you'll be glad you learned about control surfaces and sign conventions earlier on.



2.8.3 New balance of mass



Video: With sufficient skills (and lots of practice!), it is possible for a musician to produce an uninterrupted stream of air into an instrument while still continuing to breathe, a technique called circular breathing.^W Can you identify the different terms of Equation (2/28) as they apply to the saxophonist's mouth?

by David Hernando Vitores (CC-BY-SA)
<https://frama.link/vyH-cxCL>

We write our new-and-improved mass balance equation with the template provided by the Reynolds Transport Theorem, Equation (2/27). The placeholder variable B is mass m . It follows that $d B / dt$ becomes dm_{sys} / dt , which by definition is zero. Also, $b \equiv B/m = m/m = 1$ and now the Reynolds transport theorem becomes:

$$\frac{dm_{\text{sys}}}{dt} = 0 = \frac{d}{dt} \iiint_{\text{CV}} \rho \, d\mathcal{V} + \iint_{\text{CS}} \rho (\vec{V}_{\text{rel}} \cdot \vec{n}) \, dA \quad (2/28)$$

Advice from an expert

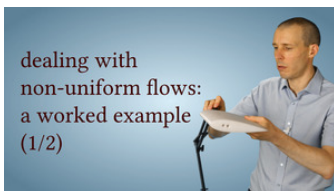
If you are thinking that equation Equation (2/28) is just a very fancy way to write the “good” mass balance equation we already wrote in chaptertwoshort (eq. eq-mass-oned), you are not completely wrong. There are two useful improvements, however:

First, having $\iint \rho V_{\perp} \, dA$ instead of just $\rho V_{\perp} A$ allows us to handle cases where the incoming/outgoing velocities are not uniform. Outside of fluid mechanics textbooks, very few flows have a nice smooth uniform outlet!

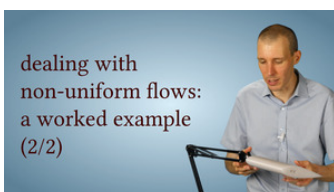
Second, there is an unsteady term dm_{CV} / dt , expressed with a cool triple integral. It is not used to solve problems in this course, but one day when you are confronted to a case where your inlet and outlet mass flows are not equal, it will save your day!



2.8.4 New balance of momentum



Learnfluidmechanics.com video 2.17:
 Non-uniform flows: an example 1/2
 by Olivier Cleyne
<https://learnfluidmechanics.com/>



Learnfluidmechanics.com video 2.18:
 Non-uniform flows: an example 2/2
 by Olivier Cleyne
<https://learnfluidmechanics.com/>

We write our new-and-improved momentum balance equation with the same template, the Reynolds Transport Theorem. The placeholder variable B is now momentum $m\vec{V}$. It follows that $d B / dt$ becomes $d(m\vec{V}_{\text{sys}}) / dt$, which by definition is the net force $\vec{F}_{\text{net}}$ applying to the system. Also, $b \equiv B/m = m\vec{V}/m = \vec{V}$ and now we get:

$$\frac{d(m\vec{V}_{\text{sys}})}{dt} = \vec{F}_{\text{net}} = \frac{d}{dt} \iiint_{\text{CV}} \rho \vec{V} \, d\mathcal{V} + \iint_{\text{CS}} \rho \vec{V} (\vec{V}_{\text{rel}} \cdot \vec{n}) \, dA \quad (2/29)$$

Let's take again some time to ponder the meaning of Equation (2/29). What could cause $\vec{F}_{\text{net}}$ to be non-zero there?

- The first term, which we could informally write as $d(m\vec{V})_{\text{CV}} / dt$, could be non-zero. This happens when the momentum inside the control volume changes. This may occur if the distribution of velocities $\vec{V}$ within the control

volume is changing, such as when the fluid in a tank sloshes back and forth against the walls.

- The last term, which we could informally write as $|\dot{m}|_2 \vec{V}_2 - |\dot{m}|_1 \vec{V}_1$, could also be non-zero. This happens when the flux of momentum entering the control volume is different from the one leaving it:
 - It may be because the mass flow $\dot{m}$ is different at inlet and outlet, even if the two velocity distributions are the same;
 - It may be because the velocities have different length, and the flow is speeding up or slowing down;
 - It may be because the velocities are aligned differently, and the flow is changing directions;
 - It may be because the velocities are non-uniform and *distributed* differently, even if their average (and thus the mass flows) are the same.

As you can see, a lot of different things may be happening at once!

2.8.5 New balance of energy

Our new-and-improved energy balance equation is used by using the template provided by the Reynolds transport theorem. At this point, we will just jump straight to the result:

$$\frac{dE_{\text{sys}}}{dt} = \dot{Q} + \dot{W} = \frac{d}{dt} \iiint_{\text{CV}} \rho e \, d\mathcal{V} + \iint_{\text{CS}} \rho e (\vec{V}_{\text{rel}} \cdot \vec{n}) \, dA \quad (2/30)$$

2.9 Limits of integral analysis

Integral analysis is an incredibly useful tool in fluid dynamics: in any given problem, it allows us to rapidly describe and calculate the main fluid phenomena at hand. The net force exerted on the fluid as it is deflected downwards by a helicopter, for example, can be calculated using just a loosely-drawn control volume and a single vector equation.

As we progress through the end-of-chapter problems, however, the limits of this method slowly become apparent. There are two of them:

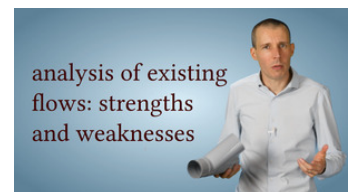
- First, we are confined to calculating the *net* effect of fluid flow. The net force, for example, encompasses the integral effect of all forces—due to pressure, shear, and gravity—applied on the fluid as it transits through the control volume. Integral analysis gives us absolutely no way of distinguishing between those sub-components. In order to do that (for example, to calculate which part of a pump's mechanical power is lost to internal viscous effects), we would need to look within the control volume.
- Second, all four of our equations in this chapter only work in one direction. The value dB/dt of any finite integral cannot be used to find which function $\rho b V_{\perp} dA$ was integrated over the control surface to obtain it. For example, there are an *infinite* number of velocity profiles which will result in a net force of +12 N. Knowing the net value of an integral, we cannot deduce the conditions which lead to it.

In practice, this is a major limitation on the use of integral analysis, because it confines us to working with large swaths of experimental data gathered at the borders of our control volumes. From the wake below the helicopter,



Video: As a person walks, the deflection of the air passing around their body can be used to sustain the flight of a paper airplane (a *walkalong glider*^W). Can you figure out the momentum flow entering and leaving a control volume surrounding the glider, and the resulting net force?

by Y:sciencetomaker (STVL)
https://youtu.be/S6JKwzK37_8



Learnfluidmechanics.com video 2.19: Strengths and weaknesses of this method

by Olivier Cleynen
<https://learnfluidmechanics.com/>



Learnfluidmechanics.com video 2.20: Control volumes: towards predicting flows

by Olivier Cleynen
<https://learnfluidmechanics.com/>

we deduce the net force; but the net force tells us nothing about the shape of the wake.

Clearly, in order to overcome these limitations, we are going to need to open up the control volume, and look at the details of the flow within – perhaps by dividing it into a myriad of sub-control volumes. This is what we set ourselves to in chapters 3 and 4, with a thundering and formidable methodology we shall call *derivative analysis*.

What ideas and equations do you think were the most important in this chapter?



Here is space reserved for a funny drawing about chapter 2

